Electromagnetic Waves :

In the absence of any source of charge or current, Maxwell's equations in free space are as follows :

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{1}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2}$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$
$$\vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{4}$$

The last two equations couple the electric and the magnetic fields. If $\vec{B}$ is time dependent, $\vec{\nabla} \times \vec{E}$ is non-zero. This implies that $\vec{E}$ is a function of position. Further, if $\partial \vec{B}/\partial t$ itself changes with time, so does $\vec{\nabla} \times \vec{E}$. In such a case $\vec{E}$ also varies with time since the $\vec{\nabla}$ operator cannot cause time variation. Thus, in general, a time varying magnetic field gives rise to an electric field which varies both in space and time. It will be seen that these coupled fields propagate in space.

We will first examine whether the equations lead to transverse waves. For simplicity, assume that the electric field has only x-component and the magnetic field only y-component. Note that we are only making an assumption regarding their directions – the fields could still depend on all the space coordinates x, y, z, in addition to time t.

Gauss's law gives

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

Since only $E_x \neq 0$, this implies

$$\frac{\partial E_x}{\partial x} = 0$$

Thus E_x is independent of x coordinate and can be written as $E_x(y, z, t)$. A similar analysis shows that B_y is independent of y coordinate and can be written explicitly as $B_y(x, z, t)$.

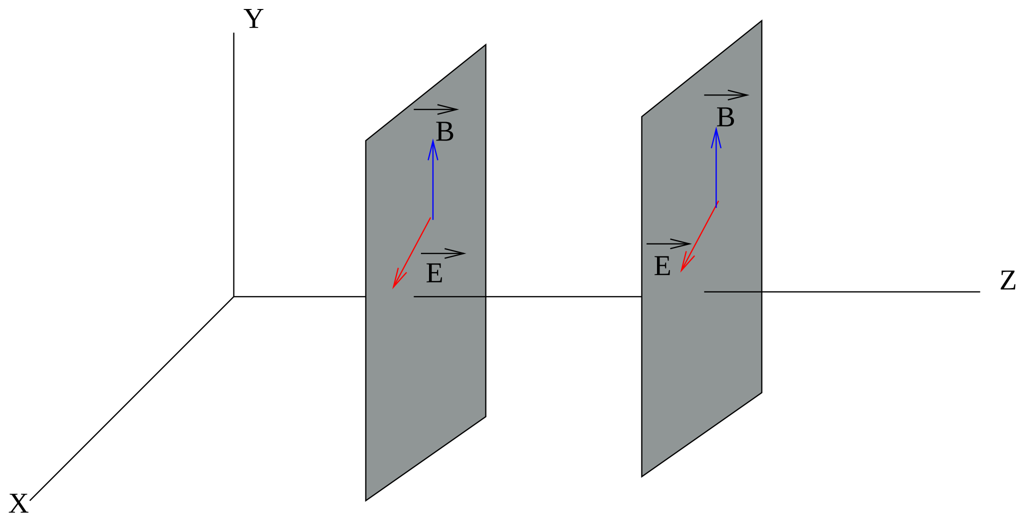
Consider now the time dependent equations eqns. (3) and (4). The curl equation for $\vec{B}$ gives, taking z-component

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{\partial E_z}{\partial t} = 0$$

Since $B_x = 0$, this gives

$$\frac{\partial B_y}{\partial x} = 0$$

showing that B_y is independent of x and hence depends only on z and t. In a similar manner we can show that E_x also depends only on z and t. Thus the fields $\vec{E}$ and $\vec{B}$ do not vary in the plane containing them. Their only variation takes place along the z-axis which is perpendicular to both $\vec{E}$ and $\vec{B}$. The direction of propagation is thus $z-$direction.



To see that propagation is really a wave disturbance, take y-component of Eqn. (3) and x-component of Eqn. (4)

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad (5)$$

$$-\frac{\partial B_y}{\partial z} = \mu_0\epsilon_0\frac{\partial E_x}{\partial t} \quad (6)$$

To get the wave equation for E_x, take the derivative of eqn. (5) with respect to z and substitute in eqn. (6) and interchange the space and time derivatives,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0\epsilon_0\frac{-\partial^2 B_y}{\partial z\partial t} = -\frac{\partial}{\partial t}\left(\frac{\partial B_y}{\partial z}\right) = \mu_0\epsilon_0\frac{\partial^2 E_x}{\partial t^2}$$

Similarly, we can show, We get

$$\frac{\partial^2 B_y}{\partial z^2} = \mu_0\epsilon_0\frac{\partial^2 B_y}{\partial t^2}$$

Each of the above equations represents a wave disturbance propagating in the z-direction with a speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

On substituting numerical values, the speed of electromagnetic waves in vacuum is 3×10^8 m/sec.
Consider plane harmonic waves of angular frequency ω and wavlength $\lambda = 2\pi/k$. We can express the waves as

$$\begin{aligned} E_x &= E_0 \sin(kz - \omega t) \\ B_y &= B_0 \sin(kz - \omega t) \end{aligned}$$

The amplitudes E_0 an B_0 are not independent as they must satisfy eqns. (5) and (6) :

$$\begin{aligned} \frac{\partial E_x}{\partial z} &= E_0 k \cos(kz - \omega t) \\ \frac{\partial B_y}{\partial t} &= -B_0 \omega \cos(kz - \omega t) \end{aligned}$$

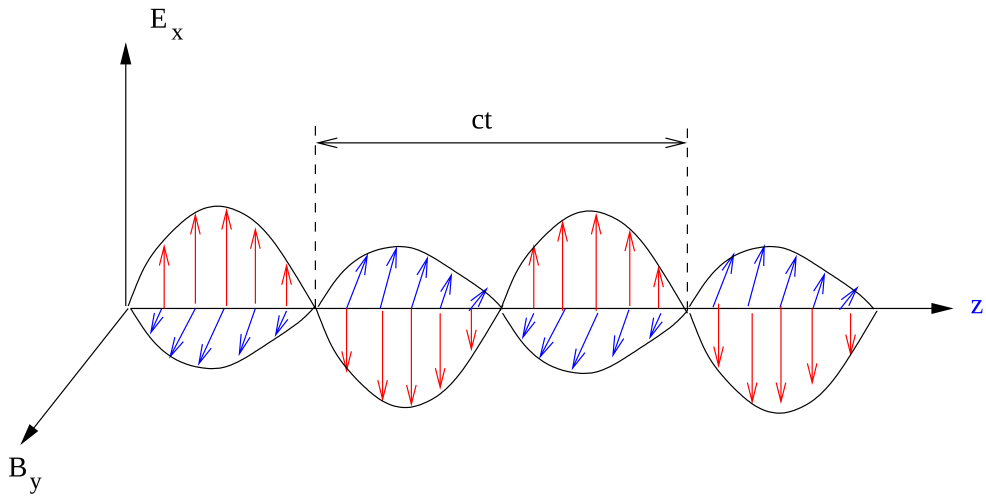
Using Eqn. (5) we get

$$E_0 k = B_0 \omega$$

The ratio of the electric field amplitude to the magnetic field amplitude is given by

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c$$

Fields $\vec{E}$ and $\vec{B}$ are in phase, reaching their maximum and minimum values at the same time. The electric field oscillates in the x-z plane and the magnetic field oscillates in the y-z plane. This corresponds to a **polarized wave**. Conventionally, the plane in which the electric field oscillates is defined as the plane of polarization. In this case it is x-z plane. The figure shows a harmonic plane wave propagating in the z-direction. Note that $\vec{E}, \vec{B}$ and the direction of propagation $\hat{k}$ form a right handed triad.



Example :
The electric field of a plane electromagnetic wave in vacuum is $E_y = 0.5\cos[2\pi \times 10^8(t - x/c)]$ V/m, $E_x = E_z = 0$. Determine the state of polarization and the direction of propagation of the wave. Determine the magnetic field.
Solution :
Comparing with the standard form for a harmonic wave

$$\begin{aligned} \omega &= 2\pi \times 10^8 \ \text{rad/s} \\ k &= 2\pi \times 10^8/c \end{aligned}$$

so that $\lambda = c/10^8 = 3$ m. the direction of propagation is x-direction. Since the electric field oscillates in x-y plane, this is the plane of polarization. Since $\vec{B}$ must be perpendicular to both the electric field direction and the direction of propagation, $\vec{B}$ has only z-component with an amplitude $B_0 = E_0/c \simeq 1.66 \times 10^{-9}$ T. Thus

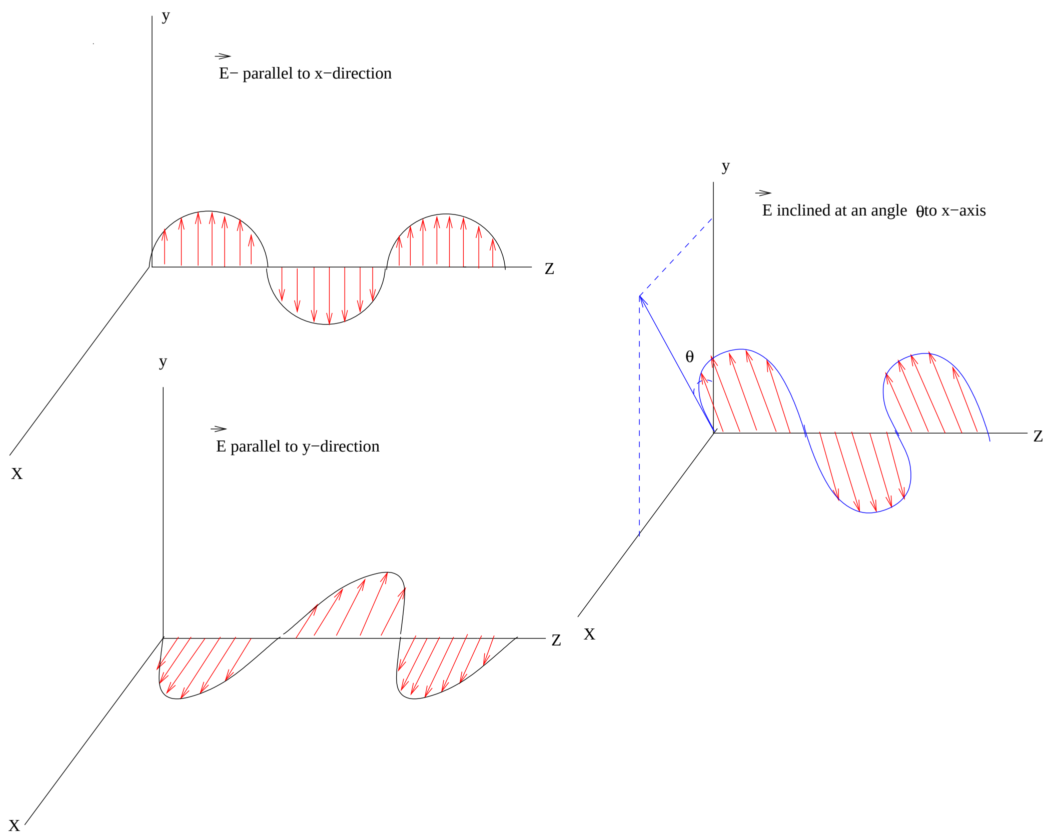
$$B_z = 1.66 \times 10^{-9}\cos[2\pi \times 10^8(t - x/c)] \ \text{T}$$

Exercise :
The magnetic field of a plane electromagnetic wave is given by

$$B_y = B_z = 10^{-8}\sin[\frac{2\pi}{3}x - 2\pi \times 10^8 t] \ \text{T}$$

Determine the electric field and the plane of polarization. (Ans. Strength of electric field is $3\sqrt{2}$ V/m)
Plane, Circular and Elliptic Polarization :

We have shown that $\vec{E}$ and $\vec{B}$ lies in a plane perpendicular to the direction of propagation, viz., the plane of polarization. This does not imply that the direction of these fields are constant in time. If it so happens that the successive directions of $\vec{E}$ (and hence $\vec{B}$) remains parallel, the electric vectors at different points in space along the direction of propagation at a given time will lie in a plane. (Equivalently, the directions of electric vectors at a given point in space at different times will be parallel). Such a situation is called a **plane polarized wave.**



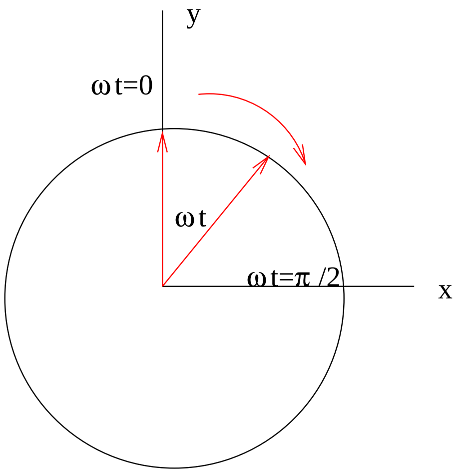
A plane polarized wave propagating in z-direction can be described by an electric field given by

$$\vec{E} =_0 \sin \omega t \hat{n} = E_0 \sin \omega t \cos \theta \hat{i} + E_0 \sin \omega t \sin \theta \hat{j}$$

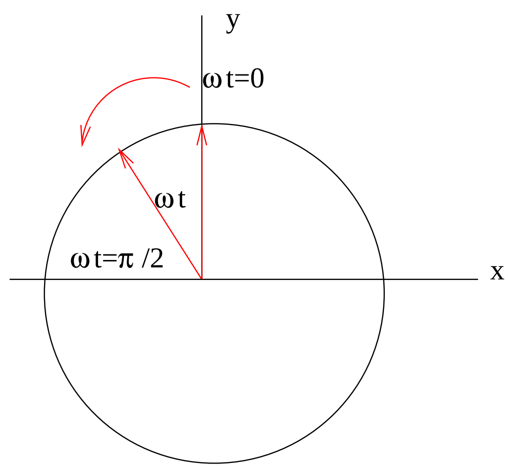
We know that because of linearity of the wave equation, any linear combination of two solutions of the wave equation is also a solution of the wave equation. This allows us to construc new states of polarization of the electromagnetic wave from the plane polarized wave. Suppose we have two plane polarized waves of equal amplitude one with the electric vector parallel to the x-direction and the other parallel to y-direction, the two waves having a phase difference of $\pi/2$. The resultant, which is also a solution of the wave equation, has

$$\begin{aligned} E_x &= E_0 \sin \omega t \\ E_y &= E_0 \cos \omega t \end{aligned}$$

The tip of the resultant electric vector $\vec{E}$ describes a circle of radius E_0, which is independent of t. The state of polarization is known as **circular polarization.** Looking along the direction of propagation if the radius vector is moving clockwise, the polarization is called **right circularly polarized** and if anticlokwise it is called left circularly polarized.



Right Circularly polarized



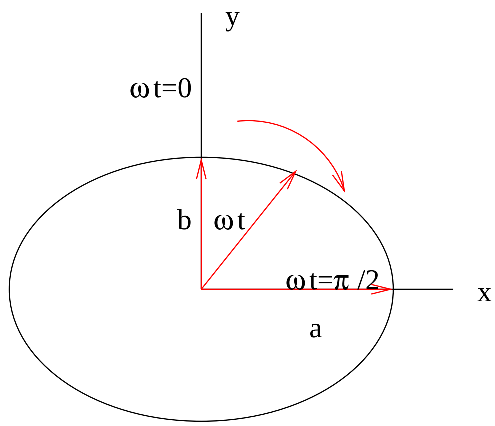
Left Circularly polarized

In the same way if the two plane polarized solutions have a phase difference of $\pi/2$ but have different amplitudes a and b, they produce what is known as elliptically polarized light. In this case, we have

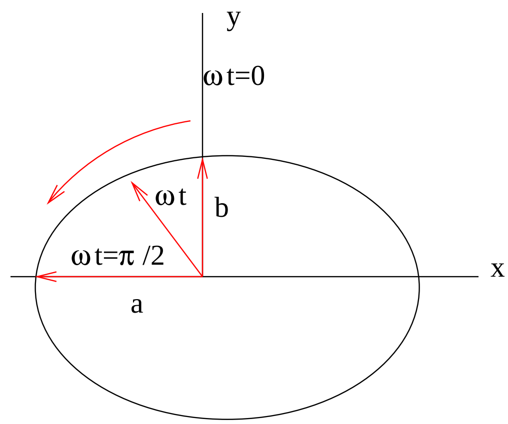
$$\begin{aligned} E_x &= a \sin \omega t \\ E_y &= b \cos \omega t \end{aligned}$$

so that the equation to the trajectory is given by

$$\frac{E_x^2}{a^2} + \frac{E_y^2}{b^2} = 1$$



Right Elliptically polarized



Left Elliptically polarized

Wave Equation in Three Dimensions :

We can obtain the wave equation in three dimensions by using eqns. (1) to (4). On taking the curl of both sides of eqn. (3), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Using the operator identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\vec{\nabla}^2 \vec{E}$$

wherein we have used $\vec{\nabla} \cdot \vec{E} = 0$, and substituting eqn. (4) we get

$$\vec{\nabla}^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

A three dimensional harmonic wave has the form $\sin(\vec{k} \cdot \vec{r} - \omega t)$ or $\cos(\vec{k} \cdot \vec{r} - \omega t)$ Instead of using the trigonometric form, it is convenient to use the complex exponential form

$$f(\vec{r}, t) = \exp(i\vec{k} \cdot \vec{r} - \omega t)$$

and later take the real or imaginary part of the function as the case may be. The derivative of $f(\vec{r}, t)$ is given as follows :

$$\frac{\partial}{\partial x} f(\vec{r}, t) = \frac{\partial}{\partial x} \exp(ik_x x + ik_y y + ik_z z - i\omega t) = ik_x f(\vec{r}, t)$$

Since

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

we have,

$$\nabla f(\vec{r},t) = i\vec{k} f(\vec{r},t)$$

In a similar way,

$$\frac{\partial}{\partial t} f(\vec{r},t) = -i\omega f(\vec{r},t)$$

Thus for our purpose, the differential operators ∇ and $\partial/\partial t$ may be equivalently replaced by

$$\begin{aligned} \frac{\partial}{\partial t} &\rightarrow -i\omega \\ \vec{\nabla} &\rightarrow i\vec{k} \end{aligned}$$

Using these, the Maxwell's equations in free space become

$$\vec{k}\cdot\vec{E} = 0 \tag{7}$$

$$\vec{k}\cdot\vec{B} = 0 \tag{8}$$

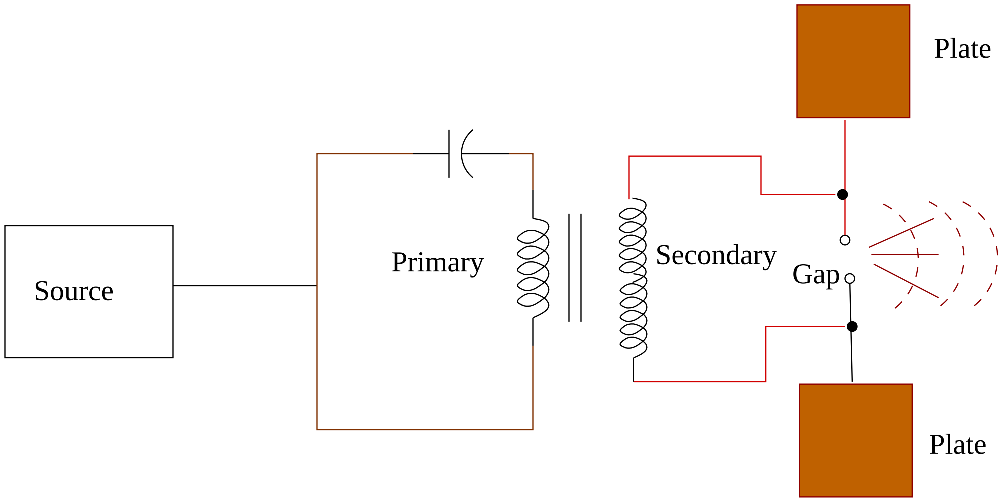
$$\vec{k}\times\vec{B} = -\mu_0\epsilon_0\vec{E} \tag{9}$$

$$\vec{k}\times\vec{E} = B \tag{10}$$

We can see that $\vec{E}, \vec{B}$ and $\vec{k}$ form a mutually orthogonal triad. The electric field and the magnetic field are perpendicular to each other and they are both perpendicular to the direction of propagation.

Generation of Electromagnetic Waves :

We have looked for solutions to Maxwell's equations in free space which does not have any charge or current source. In the presence of sources, the solutions become complicated. If ρ = constant, i.e. if $\vec{J} = 0$, we only have a steady electric field. If ρ varies uniformly with time, we have steady currents which gives us both a steady electric field as well as a magnetic field. Clearly, time varying electric and magnetic fields may be generated if the current varies with time, i.e., if the charges accelerate. Hertz confirmed the existence of electromagnetic waves in 1888 using these principles. A schematic diagram of Hertz's set up is shown in the figure.



The radiation will be appreciable only if the amplitude of oscillation of charge is comparable to the wavelength of radiation that it emits. This rules out mechanical vibration, for assuming a vibrational frequency of 1000 cycles per second, the wavelength work out to be 300 km. Hertz, therefore, made the oscillating charges vibrate with a very high frequency. The apparatus consists of two brass plates connected to the terminals of a secondary of a transformer. The primary consists of an LC oscillator circuit, which establishes charge oscillations at a frequency of $\omega = 1/\sqrt{LC}$. As the primary circuit oscillates, oscillations are set up in the secondary circuit. As a result, rapidly varying alternating potential difference is developed across the gap and electromagnetic waves are generated. Hertz was able to produce waves having wavelength of 6m. It was soon realized that irrespective of their wavelength, all electromagnetic waves travel through empty space with the same speed, viz., the speed of light.

Depending on their wavelength range, electromagnetic waves are given different names. The figure shows the electromagnetic spectrum. What is known as visible light is the narrow band of wavelength from 400 nm (blue) to 700 nm (red). To its either side are the infrared from 700 nm to 0.3 mm and the ultraviolet from 30 nm to 400 nm. Microwaves have longer wavelength than the infrared (0.3 mm to 300 mm) and the radio waves have wavelengths longer than 300 mm. The television broadcast takes place in a small range at the end of the microwave spectrum. Those with wavelengths shorter than ultraviolet are generally called *rays*. Prominent among them are x-rays with wavelengths 0.03 nm t0 30 nm and γ-rays with wavelengths shorter than 0.03 nm.

Poynting Vector :
Electromagnetic waves, like any other wave, can transport energy. The power through a unit area in a direction normal to the area is given by **Poynting vector**, given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

As $\vec{E}, \vec{B}$ and $\vec{k}$ form a right handed triad, the direction of $\vec{S}$ is along the direction of propagation. In SI units $\vec{S}$ ismeasured in watt/m^2.
The magnitude of $\vec{S}$ for the electromagnetic wave travelling in vacuum is given

by

$$S = \frac{EB}{\mu_0} = \frac{E^2}{c\mu_0}$$

where we have used the relationship between E and B in free space. For harmonic waves, we have

$$S = \frac{E_0^2}{c\mu_0}\sin^2(kx - \omega t)$$

The average power transmitted per unit area, defined as the **intensity** is given by substituting the value 1/2 for the average of the square of sine or cosine function

$$I = \frac{E_0^2}{2c\mu_0}$$

Example :
Earth receives 1300 watts per squar meter of solar energy. assuming the energy to be in the form of plane electromagnetic waves, compute the magnitude of the electric and magnetic vectors in the sunlight.
Solution :
From the expression for the average Poynting vector

$$\frac{E_0^2}{2c\mu_0} = 1300$$

which gives $E_0 = 989$ V/m. The corresponding rms value is obtained by dividing by $\sqrt{2}$, $E_{rms} = 700$ V/m. The magnetic field strength is $B_{rms} = E_{rms}/c = 2.33 \times 10^{-6}$ T.
Exercise :
A 40 watt lamp radiates all its energy isotropically. Compute the electric field at a distance of 2m from the lamp. (Ans. 30 V peak)
Radiation Pressure :
We have seen that electric field, as well as magnetic field, store energy. The energy density for the electric field was seen to be $(1/2)\epsilon_0 E^2$ and that for the magnetic field was found to be $(1/2)B^2/2\mu_0$. For the electromagnetic waves, where $E/B = c$, the total energy density is

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0} = \epsilon_0 E^2$$

where we have used $c^2 = 1/\mu_0\epsilon_0$.
In addition to carrying energy, electromagnetic waves carry momentum as well.

The relationship between energy (U) and momentum (p) is given by relatistic relation for a massless photons as $p = U/c$. Since the energy density of the electromagnetic waves is given by $\epsilon_0 E^2$, the momentum density, i.e. momentum per unit volume is

$$| p |= \frac{\epsilon_0 E^2}{c} = \epsilon_0 \mid \vec{E} \times \vec{B} \mid$$

Since the direction of momentum must be along the direction of propagation of the wave, the above can be converted to a vector equation

$$\vec{p} = \epsilon_0 \vec{E} \times \vec{B}$$

If an electromagnetic wave strikes a surface, it will thus exert a pressure. Consider the case of a beam falling normally on a surface of area A which absorbs the wave. The force exerted on the surface is equal to the rate of change of momentum of the wave. The momentum change per unit time is given by the momentum contained within a volume cA. The pressure, obtained by dividing the force by A is thus given by

$$P = cp = c\epsilon_0 EB = \epsilon_0 E^2$$

which is exactly equal to the energy density u.
If on the other hand, the surface reflects the wave, the pressure would be twice the above value.
The above is true for waves at normal incidence. If the radiation is diffuse, i.e., if it strikes the wall from all directions, it essentially consists of plane waves travelling in all directions. If the radiation is isotropic, the intensity of the wave is the same in all directions. The contribution to the pressure comes from those waves which are travelling in a direction which has a component along the normal to the surface. Thus on an average a third of the radiation is responsible for pressure. The pressure for an absorbing surface is $u/3$ while that for a reflecting surface is $2u/3$.
The existence of radiaton pressure can be verified experimentally. The curvature of a comet's tail is attributed to the radiation pressure exerted on the comet by solar radiation.

Exercise :

Assuming that the earth absorbs all the radiation that it receives from the sun, calculate he radiation pressure exerted on the earth by solar radiation. (Ans. Assuming diffuse radiation 1.33×10^{-6} N/m^2)

Wave Propagation in Matter :

Inside matter but in region where there are no sources, the relevant equations are

$$\begin{aligned}
\nabla \cdot \vec{D} &= 0 \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{\partial B}{\partial t} \\
\nabla \times \vec{H} &= \frac{\partial D}{\partial t}
\end{aligned}$$

which results in a travelling wave with speed $v = 1/\sqrt{\epsilon\mu}$ where $\vec{B} = \mu\vec{H}$ and $\vec{D} = \epsilon\vec{H}$.

Waves in a conducting Medium :

If the medium is conducting, we need to include the effect due to conduction current. The two curl equations become

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} = -\mu\frac{\partial \vec{H}}{\partial t} \\
\nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t} + \vec{J} = \epsilon\frac{\partial \vec{E}}{\partial t} + \sigma\vec{E}
\end{aligned}$$

where we have used Ohm's law as another constitutive relation. Thus, we have,

$$\nabla \times (\nabla \times \vec{E}) = -\mu\frac{\partial}{\partial t}(\nabla \times \vec{H})$$

Using the expansion for $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 E$, we get

$$\frac{1}{\epsilon}\nabla(\nabla \cdot \vec{D}) - \nabla^2\vec{E} = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial E}{\partial t}$$

It may be noted that though $\nabla \cdot \vec{D}$ term equals ρ_f, free charges, if they exist in a conductor soon depletes. This may be seen from the equation of continuity. Using

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}$$

and using Ohm's law one can see that if at time $t = 0$ there exists some free charge ρ_f^0, the charge drops exponentially to 1/e th of its value after a time ϵ/σ which is very small for conductors. Thus the wave equation that we have is

$$\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial E}{\partial t}$$

and a similar equation for $\vec{H}$, i.e.

$$\nabla^2 \vec{H} = \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} + \mu\sigma \frac{\partial t}{\partial t}$$

In practice, most generators produce voltages and currents which vary sinusoidally. Corresponding electric and magnetic fields also vary simusoidally with time. This implies $\partial E/\partial t \longrightarrow -i\omega E$ and $\partial^2 E/\partial t^2 \longrightarrow -\omega E$. The wave equation becomes

$$\nabla^2 \vec{E} = -\mu\epsilon\omega^2 \vec{E} + i\mu\sigma\omega E \equiv \gamma^2 E$$

where

$$\gamma^2 = -\mu\epsilon\omega^2 + i\mu\sigma\omega$$

is a complex constant. We may write $\gamma = \alpha + i\beta$, squaring and equating it to the expression for γ^2, we can see that α and β have the same sign. When we take the square root of γ^2 and write it as γ we assume that α and β are positive. One can explicitly show that

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1 \right)}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1 \right)}$$

We have seen that in Maxwell's equation, $\sigma\vec{E}$ is conduction current density while the term $i\omega\epsilon\vec{E}$ is the displacement current density. This suggest that the value of the ratio $\sigma/\omega\epsilon$ is a good measure to divide whether a material is a dielectric or a metal. For good conductors this ratio is much larger than 1 over the entire radio frequency spectrum. For instance, at a frequency as high as 30, 000 MHz, Cu has $\sigma/\omega\epsilon \simeq 10^8$ while for the same frequency this ratio is 0.0002 for mica.
For good conductors, we take $\sigma/\omega\epsilon \gg 1$. We have, in this limit

$$\alpha = \beta = \sqrt{\frac{\omega\sigma\mu}{2}}$$

The velocity of the wave in the conductor is

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

The wave is greatly attenuated inside a conductor. **Depth of penetration** or **skin depth** δ is defined as the distance at which an eletromagnetic wave would have attenuated to 1/e of its value on the surface.

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

For conductors like copper this distance is typically less than a fraction of a millimeter.